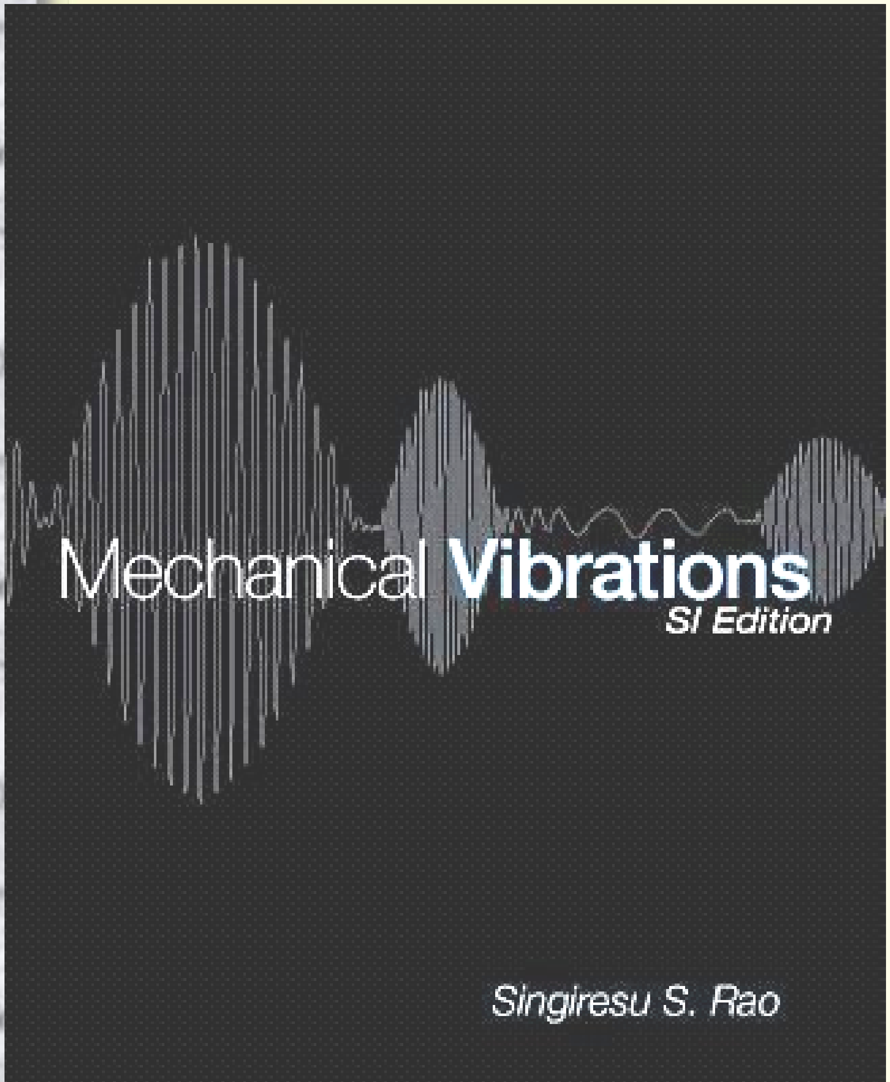

Mechanical Vibrations

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SI Edition

PEARSON Education Asia

Free Vibration of SDOF

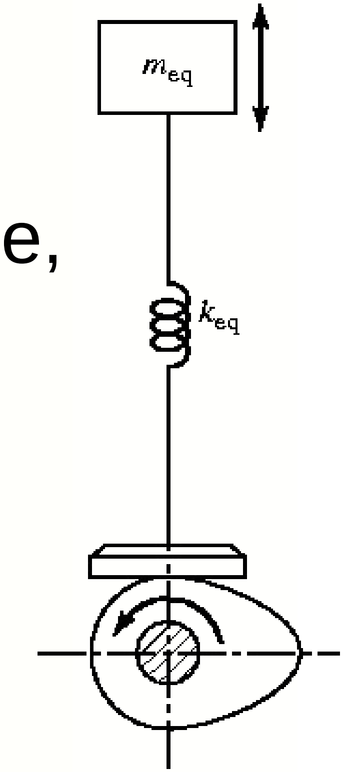
Chapter Outline

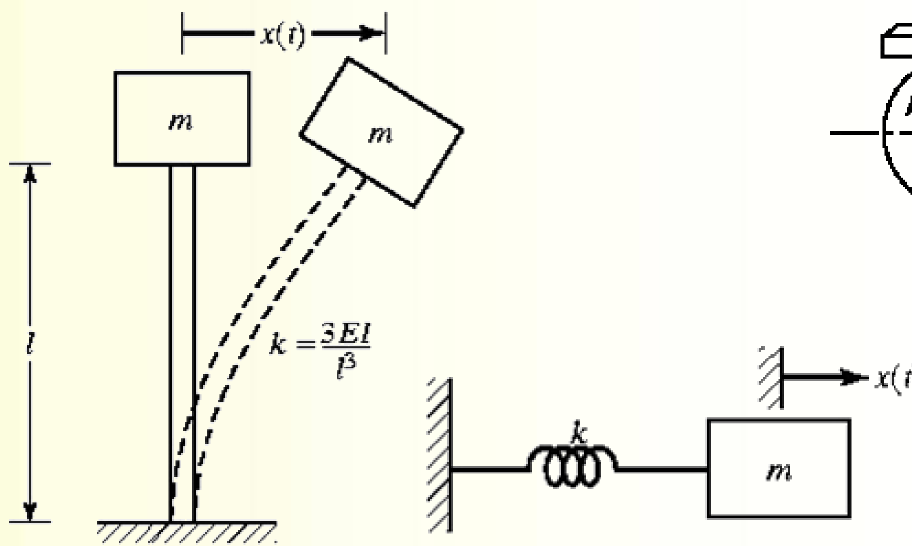
2.1 Introduction

- Free Vibration occurs when a system oscillates only under an initial disturbance with no external forces acting after the initial disturbance
- Undamped vibrations result when amplitude of motion remains constant with time (e.g. in a vacuum)
- Damped vibrations occur when the amplitude of free vibration diminishes gradually overtime, due to resistance offered by the surrounding medium (e.g. air)

2.1 Introduction

- Several mechanical and structural systems can be idealized as single degree of freedom systems, for example, the mass and stiffness of a system





(a) Idealization of the tall structure

(b) Equivalent spring-mass system

Equation of motion

- Newton's second law
- Principle of Conservation of Energy
- Rayleigh method
- Lagrange equation
- D'Alembert principle

Free Vibration of an Undamped Translational System

Equation of Motion Using Newton's Second Law of Motion:

If mass *m* is displaced a distance $\vec{x}(t)$ when acted upon by a resultant force $\vec{F}(t)$ in the same direction,

$$\vec{F}(t) = \frac{d}{dt}\left(m \frac{d\vec{x}(t)}{dt} \right)$$

If mass *m* is constant, this equation reduces to

$$\vec{F}(t) = m \frac{d^2\vec{x}(t)}{dt^2} = m\ddot{\vec{x}}$$

where $\ddot{\vec{x}} = \frac{d^2\vec{x}(t)}{dt^2}$ is the acceleration of the mass

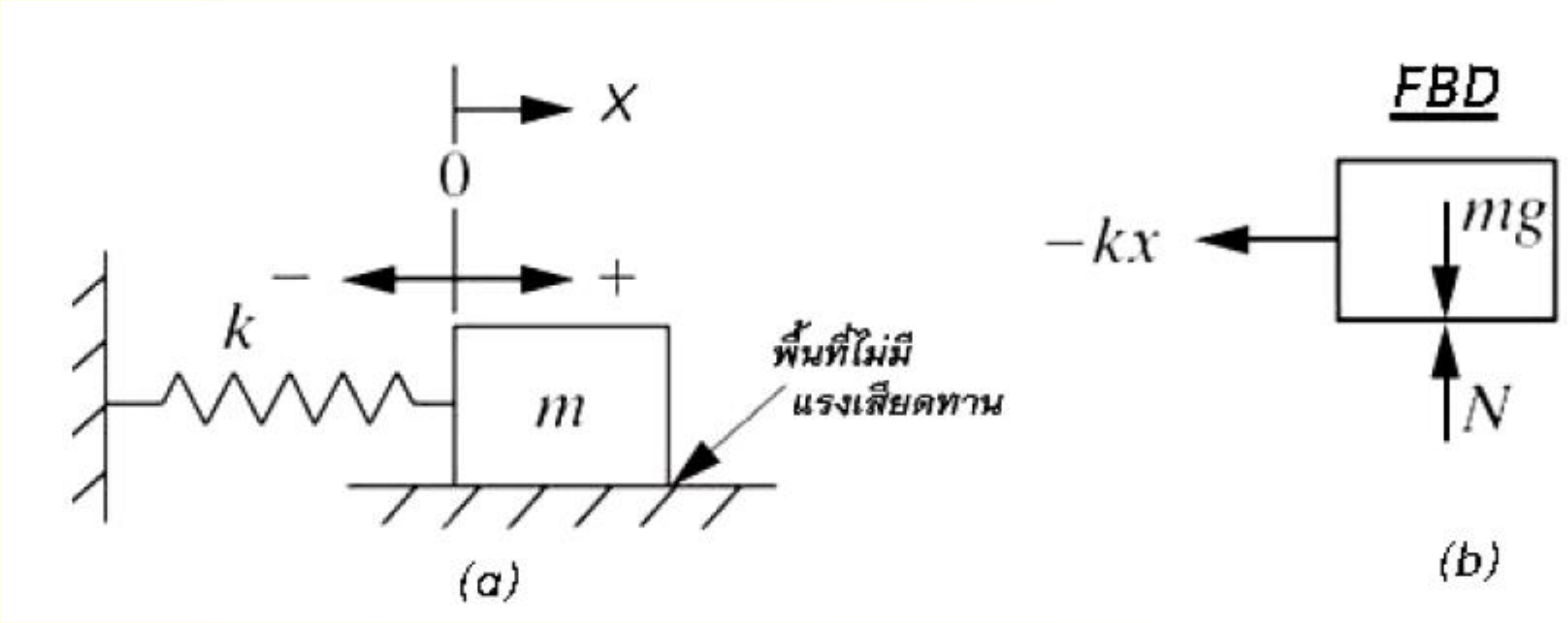
Free Vibration of an Undamped Rotational System

For a rigid body undergoing rotational motion, Newton's Law gives

$$\vec{M}(t) = J\ddot{\vec{\theta}}$$

where $\vec{M}$ is the resultant moment acting on the body and $\vec{\theta}$ and $\ddot{\vec{\theta}} = d^2\theta(t)/dt^2$ are the resulting angular displacement and angular acceleration, respectively.

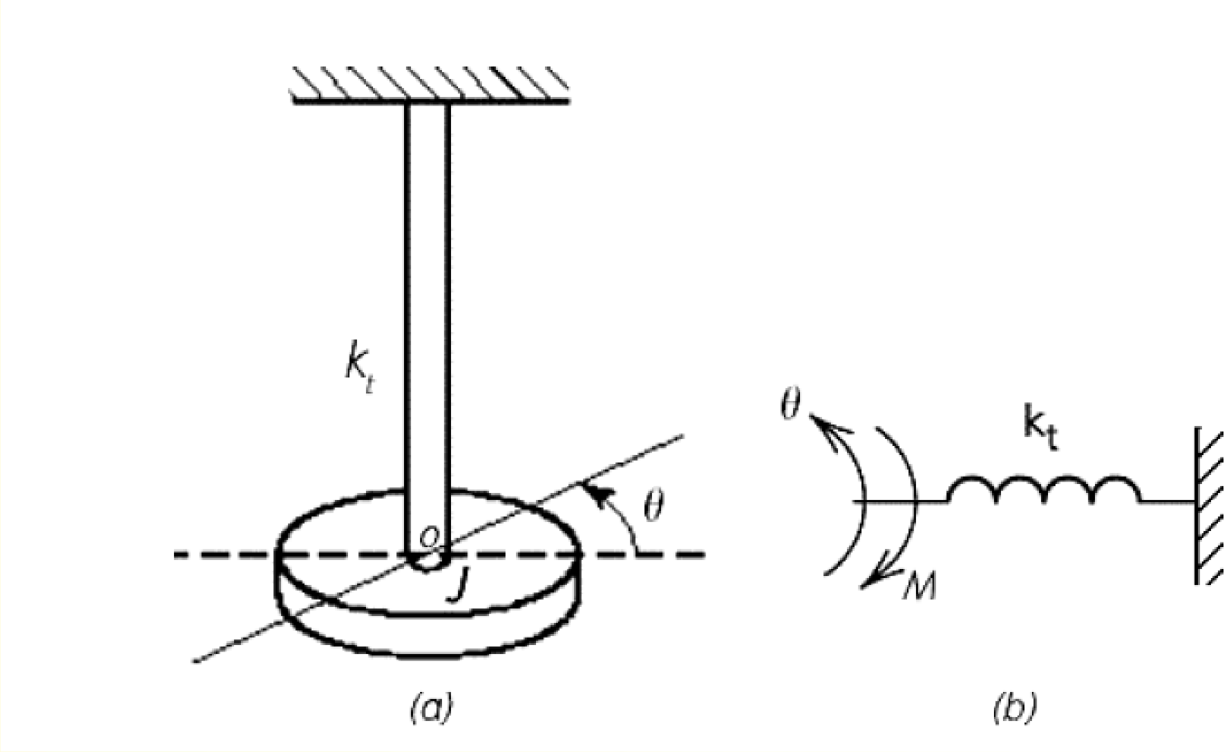
Newton's second law: SDOF



$$\sum \vec{F}(t) = m\ddot{\vec{x}}$$

Undamped Translational System

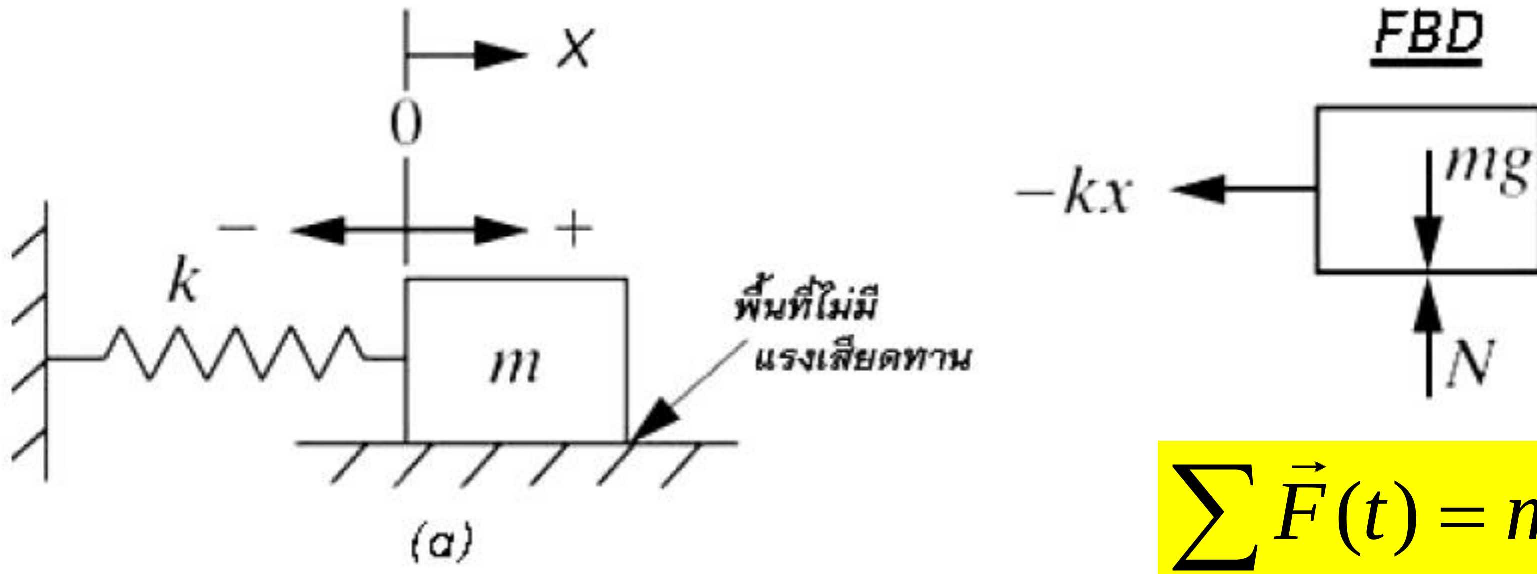
Newton's second law: SDOF



$$\sum \vec{M}(t) = J\ddot{\vec{\theta}}$$

Undamped Rotational System

Newton's second law: SDOF



$$\sum \vec{F}(t) = m\ddot{\vec{x}}$$

$$-kx = m\ddot{x}$$

Equation of motion(free vibration of SDOF)

$$m\ddot{x} + kx = 0$$

Undamped free vibration of SDOF

$$m\ddot{x} + kx = 0$$

Laplace transform

$$m\left[s^2X(s) - sx_0 - v_0\right] + kX(s) = 0$$

Natural frequency of $\omega_n = \sqrt{k/m}$

$$\left[s^2X(s) - sx_0 - v_0\right] + \omega_n^2X(s) = 0$$

$$X(s) = \frac{sx_0 + v_0}{s^2 + \omega_n^2} = x_0\left(\frac{s}{s^2 + \omega_n^2}\right) + \frac{v_0}{\omega_n}\left(\frac{\omega_n}{s^2 + \omega_n^2}\right)$$

Undamped free vibration of SDOF

Inverse Laplace, Solution of the governing equation

$$x(t) = x_0 \cos \omega_n t + \frac{v_0}{\omega_n} \sin \omega_n t \qquad \textbf{(1)}$$

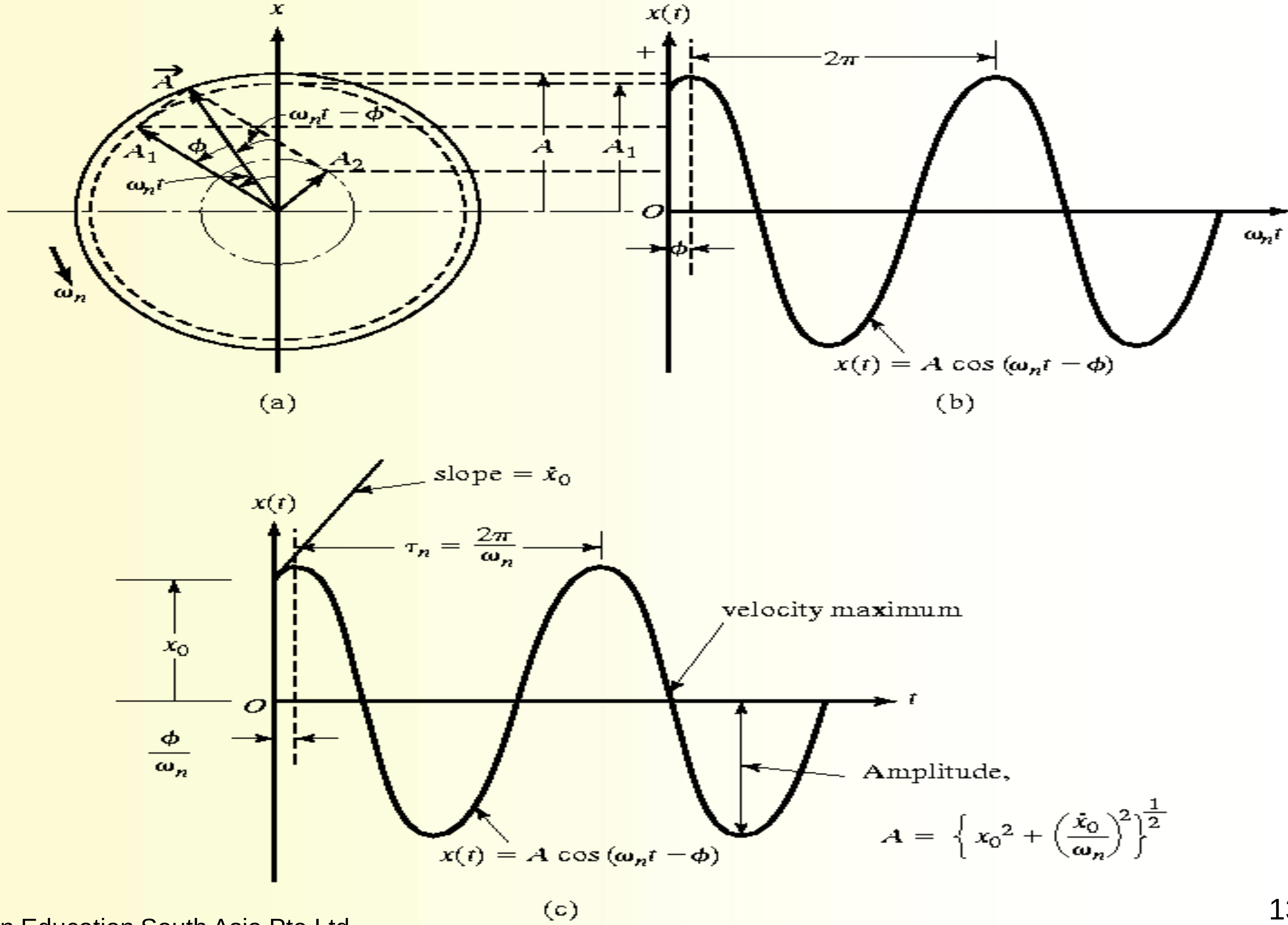
or

$$x(t) = A \sin(\omega_n t + \phi) \qquad \textbf{(2)}$$

where

$$A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2},\ \phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)$$

Undamped free vibration of SDOF



Principle of Conservation of Energy

A system is said to be conservative if no energy is lost due to friction or energy-dissipating nonelastic members.

If no work is done on the conservative system by external forces, the total energy of the system remains constant. Thus the principle of conservation of energy can be expressed as:

$$T + U = \text{constant}$$

or

$$\frac{d}{dt}(T + U) = 0$$

Principle of Conservation of Energy: SDOF

The kinetic and potential energies are given by:

$$T = \frac{1}{2} m \dot{x}^2 \quad \text{and} \quad U = \frac{1}{2} k x^2$$

$$T + U = \text{constant}$$

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \text{constant}$$

$$\frac{d}{dt}(T + U) = 0$$

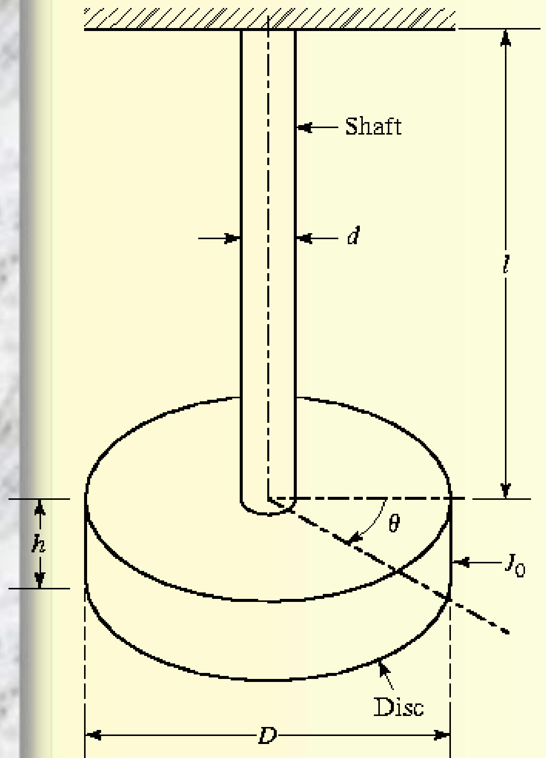
$$\frac{1}{2} m (2\dot{x}) \ddot{x} + \frac{1}{2} k (2x) \dot{x} = 0$$

Equation of motion(free vibration of SDOF)

$$m\ddot{x} + kx = 0$$

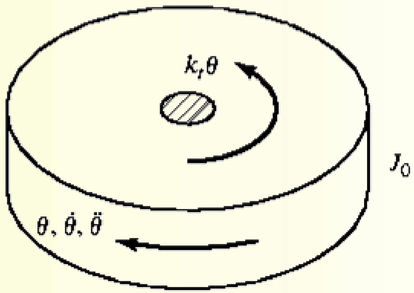
Free Vibration of an Undamped Torsinal System

Torsional Spring Constant: $$k_t = \frac{\pi G d^4}{32 l}$$



(a)

For a rigid body undergoing rotational motion, Newton's Law give



(b)

$$\sum \vec{M}(t) = J\ddot{\vec{\theta}}$$

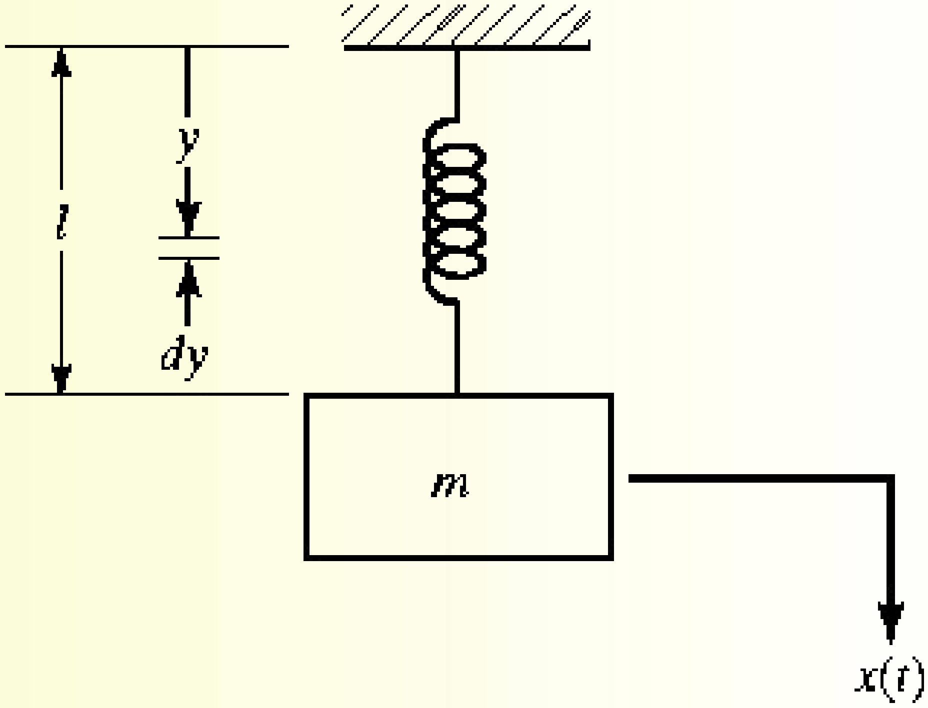
$$-k_t\theta = J\ddot{\vec{\theta}}$$

Equation of motion(free vibration of SDOF)

$$J\ddot{\vec{\theta}} + k_t\theta = 0$$

Example 2.8
Effect of Mass on ω_n of a Spring

Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown in the figure below.



Example 2.8 Solution

The kinetic energy of the spring element of length *dy* is

$$dT_s = \frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y\dot{x}}{l}\right)^2 \qquad \text{(E.1)}$$

where m_s is the mass of the spring. The total kinetic energy of the system can be expressed as

$$T = \text{ kinetic energy of mass } (T_m) + \text{ kinetic energy of spring } (T_s)$$

$$= \frac{1}{2}m\dot{x}^2 + \int_{y=0}^{l} \frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y^2\dot{x}^2}{l^2}\right)$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2 \qquad \text{(E.2)}$$

Example 2.8 Solution

The total potential energy of the system is given by

$$U = \frac{1}{2}kx^2 \qquad \text{(E.3)}$$

$$T + U = \text{constant}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant}$$

$$\frac{d}{dt}(T + U) = 0$$

$$m\ddot{x} + \frac{m_s}{3}\ddot{x} + kx = 0$$

$$\left(m + \frac{m_s}{3}\right)\ddot{x} + kx = 0$$

Example 2.8 Solution

we obtain the expression for the natural frequency:

$$\omega_n = \left(\frac{k}{m + \frac{m_s}{3}} \right)^{1/2} \qquad \text{(E.7)}$$

Thus the effect of the mass of spring can be accounted for by adding one-third of its mass to the main mass.

Free Vibration with Viscous Damping: SDOF

- Damping force: $$F = -c\dot{x}$$

where *c* = damping constant

Newton's law yields the equation of motion:

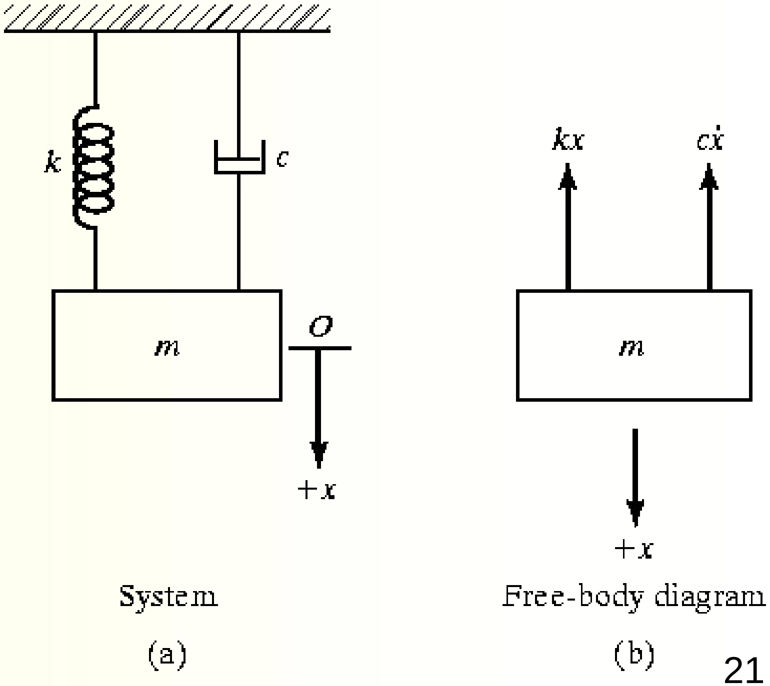
$$m\ddot{x} = -c\dot{x} - kx$$

or $$m\ddot{x} + c\dot{x} + kx = 0$$

Characteristic equation

$$ms^2 + cs + k = 0$$

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$



Free Vibration with Viscous Damping: SDOF

- Critical Damping Constant and Damping Ratio:

The critical damping c_c is defined as the value of the damping constant c for which the radical in becomes zero:

$$c^2 - 4mk = 0$$

or

$$c_c = 2\sqrt{mk} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

The damping ratio ζ is defined as:

$$\zeta = \frac{c}{c_c}$$

Free Vibration with Viscous Damping: SDOF

Assuming that $\zeta \neq 0$, consider the following 3 cases:

Case1. ***Underdamped system***

$$(0 < \zeta < 1 \quad \text{or} \quad c < c_c \ )$$

For this condition, *Poles* is negative and the roots are:

$$s_1 = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n + i\omega_d$$
$$s_2 = -\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n - i\omega_d$$

$$i = \sqrt{-1}$$

Damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Free Vibration with Viscous Damping: SDOF

and the solution can be written in different forms:

$$
\begin{aligned}
x(t) &= C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t} \\
&= e^{-\zeta\omega_n t}\left\{C_1' \cos\omega_d t + C_2' \sin\omega_d t\right\} \\
&= X e^{-\zeta\omega_n t} \sin\left(\omega_d t + \varphi\right) \\
&= X_0 e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \varphi_0\right)
\end{aligned} \tag{2.70}
$$

where *(C'_1, C'_2), (X, Φ),* and *(X_0, Φ_0)* are arbitrary constants to be determined from initial conditions.

Free Vibration with Viscous Damping: SDOF

For the initial conditions at *t = 0*,

$$C_1' = x_0 \text{ and } C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \qquad (2.71)$$

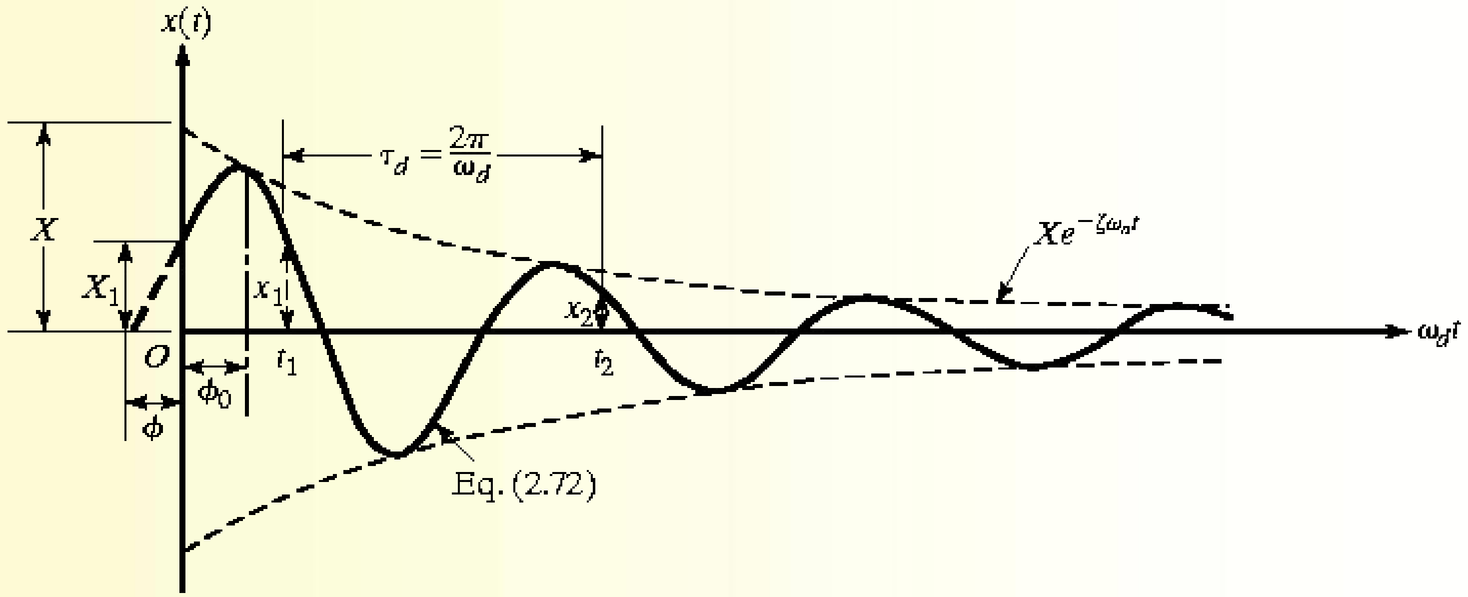
and hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\omega_d t\right\}$$

Eq. describes a damped harmonic motion. Its amplitude decreases exponentially with time, as shown in the figure below.

Free Vibration with Viscous Damping: SDOF

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\omega_d t\right\}$$



Underdamped Solution

Free Vibration with Viscous Damping: SDOF

Case2. ***Critically damped system*** $(\zeta = 1 \text{ or } c = c_c \text{ or } c/2m = \sqrt{k/m})$

In this case, the two roots(Poles) are:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

Due to repeated roots, the solution of Eq.(2.59) is given by

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{2.78}$$

Application of initial conditions gives:

$$C_1 = x_0 \text{ and } C_2 = \dot{x}_0 + \omega_n x_0 \tag{2.79}$$

Thus the solution becomes:

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0\right)t\right]e^{-\omega_n t} \tag{2.80}$$

Free Vibration with Viscous Damping: SDOF

Case2. ***Critically damped system***

$$(\zeta = 1 \quad \text{or} \quad c = c_c)$$

In this case, the two roots are:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

Due to repeated roots, the solution of critical damped system is given by

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

Application of initial conditions gives:

$$C_1 = x_0 \text{ and } C_2 = \dot{x}_0 + \omega_n x_0$$

Thus the solution becomes:

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0 \right) t \right] e^{-\omega_n t}$$

Free Vibration with Viscous Damping: SDOF

Case3. ***Overdamped system*** $(\zeta > 1 \text{ or } c > c_c \text{ or } c/2m > \sqrt{k/m})$

The roots are real and distinct and are given by:

$$s_1 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right) < 0$$

$$s_2 = \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right) < 0$$

In this case, the solution Eq.(2.69) is given by:

$$x(t) = C_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t} + C_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)t}$$

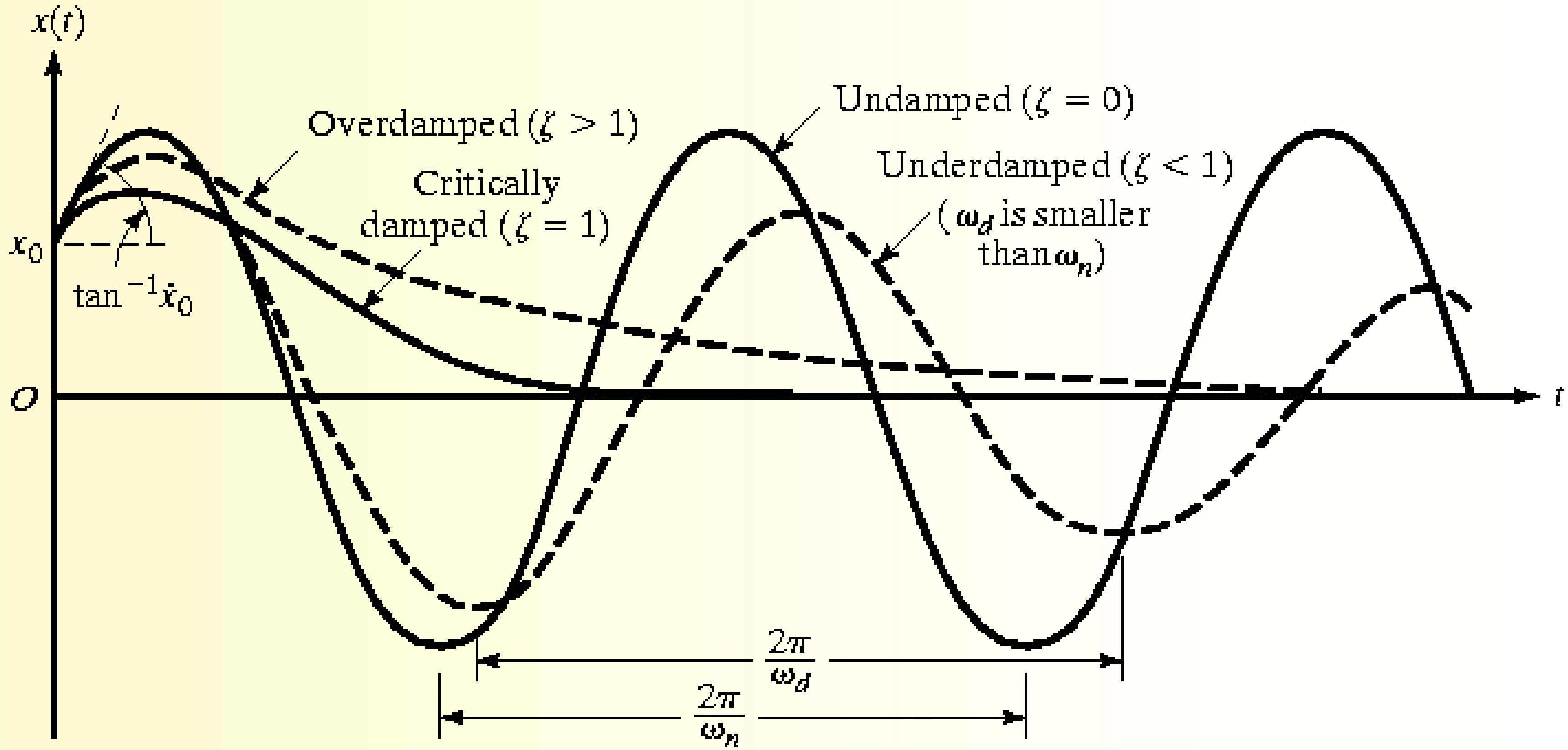
For the initial conditions at *t = 0*,

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_1 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

Free Vibration with Viscous Damping: SDOF

Since $e^{-\omega_n t} \to 0$ as $t \to \infty$, the motion will eventually diminish to zero, as indicated in the figure below.

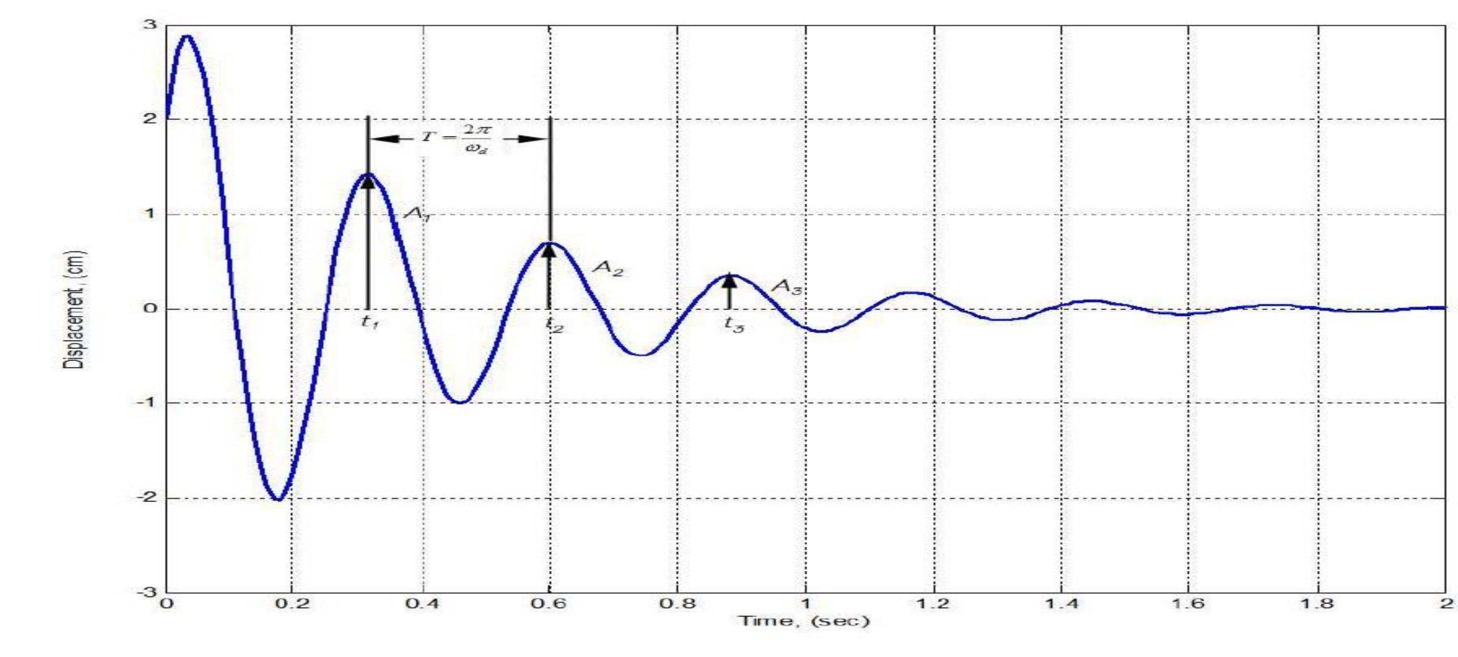


Logarithmic Decrement: Underdamped system

Solution of Underdamped free vibration:

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \varphi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \varphi_0)}$$

$$= \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+\tau_d)}} = e^{\zeta\omega_n \tau_d}$$



Logarithmic Decrement: Underdamped system

The logarithmic decrement can be obtained

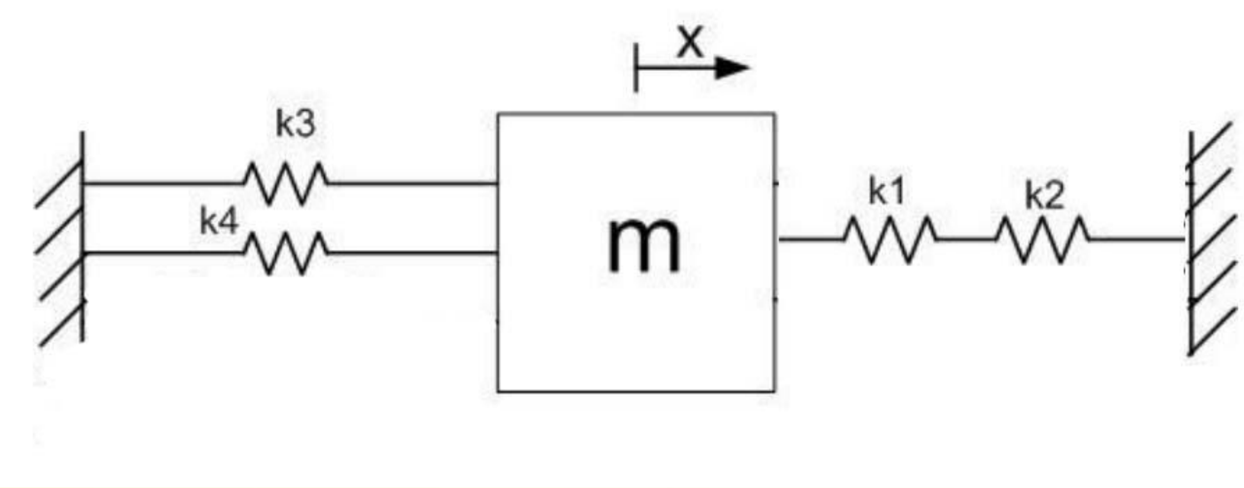
$$\delta = \ln \frac{x_1}{x_2} = \zeta \omega_n \tau_d = \zeta \omega_n \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d} \cdot \frac{c}{2m}$$

Hence,

$$\boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}}$$

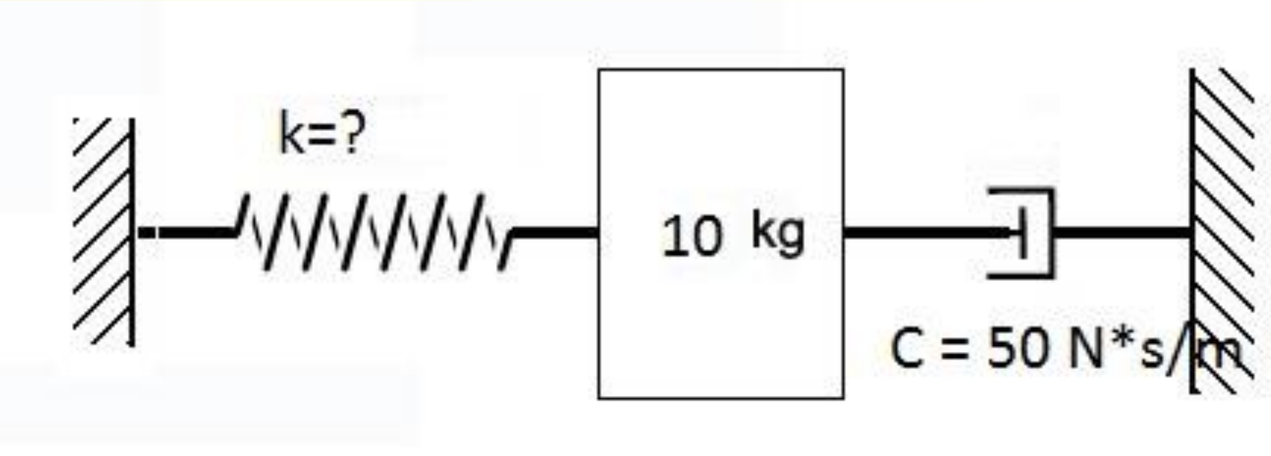
Home work - 1

Determine the differential equation of motion and the equivalent stiffness for the system (N/m) of Figure



Home work - 2

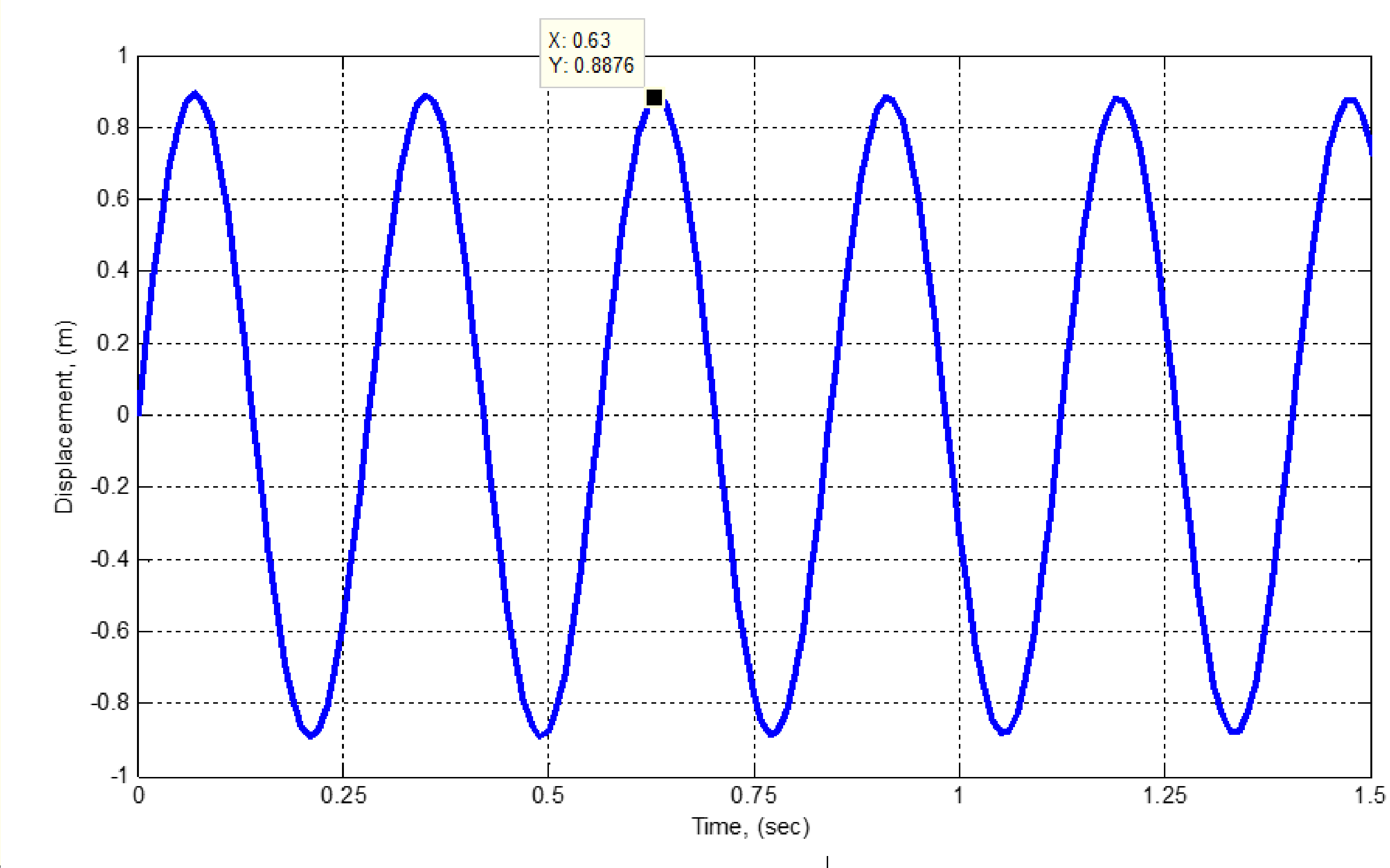
For free vibration of underdamped system($\zeta = 0.8$) as shown in Figure. Determine stiffness of system



Home work - 3

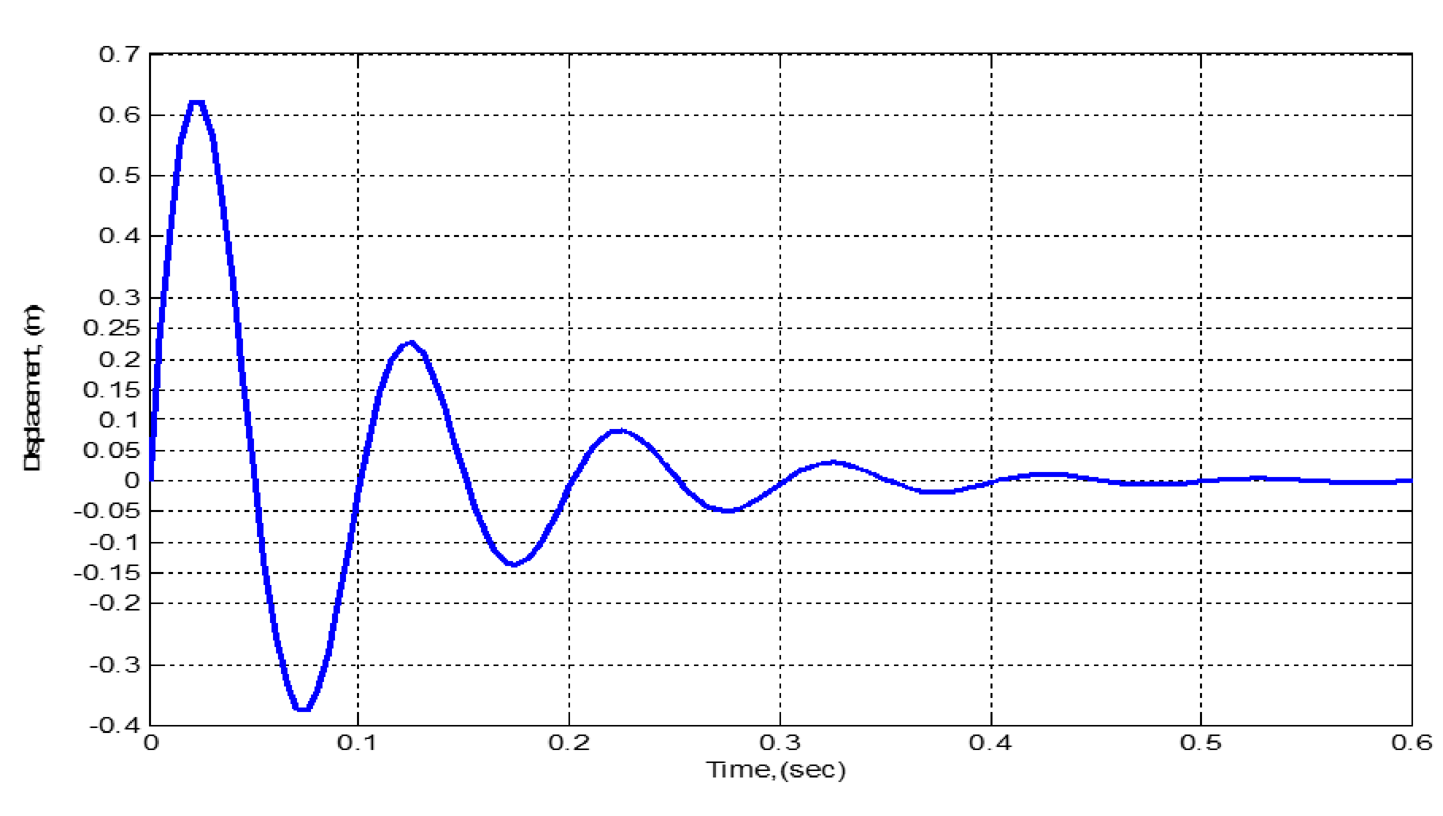
The free response of the undamped system of Figure. Determine

- Natural frequency, Initial velocity and Initial displacement



Home work - 4

The free response of the underdamped system in Figure. Calculate the damping ratio, logarithm decrement



Chapter summaries

- Equation of motion for SDOF undamped system

- Equation of motion for SDOF damped system

Chapter summaries

- Undamped free vibration response

- Overdamped free vibration response

- Critically damped free vibration response

Chapter summaries

- Underdamped free vibration response

- Logarithm decrement